

UCSBTH-92-40

MMC-M-2

October 1992

Decoherence Functional and Probability Interpretation

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ABSTRACT

We confirm that the diagonal elements of the Gell-Mann and Hartle's decoherence functional are equal to the relative frequencies of the results of many identical experiments, when a set of alternative histories decoheres. We consider both cases of the pure and mixed initial states.

§ 1. Introduction

If quantum mechanics is the fundamental theory of physics, the entire universe should also be described quantum mechanically. Recently Gell-Mann and Hartle [1] generalized the quantum theory using the concept of coarse graining and decoherence. Similar frameworks were constructed primarily by Griffiths [2] and Omnès [3], and Yamada and Takagi [4] constructed independently the similar framework. They showed that, when a set of alternative histories decoheres, the diagonal elements of the decoherence functional satisfy the mathematical properties of probabilities, so they regarded these as physical probabilities.

On the other hand, Everett [5] and others [6-12] discussed that probability interpretation of the quantum theory can be explained by the fundamental framework of the theory itself, though a “measure” in Hilbert space is introduced. They considered the relative frequencies of the results of many (N) identical experiments. They showed that the absolute squares which are identified with probabilities in ordinary quantum mechanics are equal to the relative frequencies, when $N \rightarrow \infty$.

In this paper we confirm that the diagonal elements of the Gell-Mann and Hartle’s decoherence functional are equal to the relative frequencies, when a set of alternative histories decoheres and $N \rightarrow \infty$. We consider the pure state case in §2 and the mixed state case in §3.

§ 2. Pure State Case

In order to consider probability interpretation we need an ensemble of identical systems. In this section an individual system is a pure state $|\psi\rangle$, and $|\psi\rangle$ is normalized to unity, $\langle\psi|\psi\rangle = 1$. Suppose we have N identical systems, so that the total system is described by the state vector,

$$|\Psi\rangle = |\psi\rangle^N = |\psi\rangle \cdots |\psi\rangle \quad (N \text{ terms}) . \quad (1)$$

Following Gell-Mann and Hartle [1] let us consider histories,

$$C_\alpha = P_{\alpha_n}^n(t_n) \cdots P_{\alpha_1}^1(t_1) . \quad (2)$$

Here $P_{\alpha_k}^k(t_k)$ ($k = 1, \dots, n$) are projection operators and satisfy

$$\sum_{\alpha} P_{\alpha}^k(t) = 1 \quad , \quad P_{\alpha}^k(t) P_{\beta}^k(t) = \delta_{\alpha\beta} P_{\alpha}^k(t) . \quad (3)$$

In $P_\alpha^k(t)$, k labels the set, α the particular alternative, and t its time.

The decoherence functional can be written as

$$\begin{aligned} D(C_{\alpha'}, C_\alpha) &= \text{Tr} [C_{\alpha'} \rho C_\alpha^\dagger] \\ &= \text{Tr} \left[P_{\alpha'_n}^n(t_n) \cdots P_{\alpha'_1}^1(t_1) \rho P_{\alpha_1}^1(t_1) \cdots P_{\alpha_n}^n(t_n) \right] , \end{aligned} \quad (4)$$

where ρ is the initial density matrix,

$$\rho = |\psi\rangle\langle\psi| . \quad (5)$$

A set of histories, $C_\alpha, C_{\alpha'}$ is said to decohere, when

$$D(C_{\alpha'}, C_\alpha) = 0 \quad (\text{for any } \alpha'_k \neq \alpha_k) . \quad (6)$$

In the following discussion we assume that the set of histories decoheres. From Eqs. (4)–(6) we obtain

$$\begin{aligned} D(C_{\alpha'}, C_\alpha) &= \sum_{\psi'} \langle \psi' | C_{\alpha'} | \psi \rangle \langle \psi | C_\alpha^\dagger | \psi' \rangle \\ &= \langle \psi | C_\alpha^\dagger C_{\alpha'} | \psi \rangle \\ &= \delta_{\alpha\alpha'} \langle \psi | C_\alpha^\dagger C_\alpha | \psi \rangle , \\ \text{with } \delta_{\alpha\alpha'} &= \prod_{k=1}^n \delta_{\alpha_k \alpha'_k} . \end{aligned} \quad (7)$$

Denote the diagonal element of decoherence functional as $P[C_\alpha]$, that is

$$P[C_\alpha] = D(C_\alpha, C_\alpha) . \quad (8)$$

Starting from $\sum_\beta \langle \psi | C_\beta^\dagger C_\alpha | \psi \rangle$ or $\sum_\beta \langle \psi | C_\alpha^\dagger C_\beta | \psi \rangle$, we can see

$$P[C_\alpha] = \langle \psi | C_\alpha^\dagger C_\alpha | \psi \rangle = \langle \psi | C_\alpha | \psi \rangle = \langle \psi | C_\alpha^\dagger | \psi \rangle . \quad (9)$$

It is easy to see that $P[C_\alpha]$ satisfy the axiom of mathematical probability:

$$P[C_\alpha] \geq 0, \quad \sum_\alpha P[C_\alpha] = 1, \quad P[C_\alpha + C_\beta] = P[C_\alpha] + P[C_\beta] \quad (\alpha \neq \beta) . \quad (10)$$

We will show that these $P[C_\alpha]$ are equal to the relative frequencies of the results of many (N) identical experiments.

Consider that histories $C_{\alpha^I} = P_{\alpha_n^I}^n(t_n) \cdots P_{\alpha_1^I}^1(t_1)$ ($I = 1, \dots, N$) are those of N identical systems, but they may not be same histories. Here $P_{\alpha_k^I}^k(t_k)$ ($k = 1, \dots, n$) act on I th factor of Eq. (1), upper indices of α distinguish the individual system, and lower indices distinguish the time slice.

Let us define the relative frequency* of α in the sequence $\alpha^1, \dots, \alpha^N$ by

$$f_\alpha(\alpha^1, \dots, \alpha^N) = \frac{1}{N} \sum_{I=1}^N \delta_{\alpha \alpha^I} . \quad (11)$$

And let us define

$$\delta(\alpha^1, \dots, \alpha^N) = \sum_{\alpha} [f_\alpha(\alpha^1, \dots, \alpha^N) - P[C_\alpha]]^2 , \quad (12)$$

which measures the degree to which the sequence $\alpha^1, \dots, \alpha^N$ deviates from a random sequence with weights $P[C_\alpha]$.

We write

$$|\alpha\rangle = \frac{C_\alpha |\psi\rangle}{\sqrt{\langle \psi | C_\alpha | \psi \rangle}} , \quad (13)$$

and we obtain from Eqs. (2), (3), (7), (9)

$$\begin{aligned} \langle \alpha | \beta \rangle &= \delta_{\alpha\beta} , \\ |\psi\rangle &= \sum_{\alpha} \langle \alpha | \psi \rangle |\alpha\rangle , \\ |\langle \alpha | \psi \rangle|^2 &= P[C_\alpha] . \end{aligned} \quad (14)$$

With these ortho-normal vectors $|\alpha\rangle$ we can expand the total wave function as

$$|\Psi\rangle = |\psi\rangle^N = \sum_{\alpha^1 \dots \alpha^N} \langle \alpha^1 | \psi \rangle \cdots \langle \alpha^N | \psi \rangle |\alpha^1\rangle \cdots |\alpha^N\rangle . \quad (15)$$

Let ϵ be an arbitrarily small positive number and let us define

$$\begin{aligned} |\Psi_N^\epsilon\rangle &= \sum_{\substack{\alpha^1 \dots \alpha^N \\ \delta(\alpha^1 \dots \alpha^N) < \epsilon}} \langle \alpha^1 | \psi \rangle \cdots \langle \alpha^N | \psi \rangle |\alpha^1\rangle \cdots |\alpha^N\rangle , \\ |\chi_N^\epsilon\rangle &= \sum_{\substack{\alpha^1 \dots \alpha^N \\ \delta(\alpha^1 \dots \alpha^N) \geq \epsilon}} \langle \alpha^1 | \psi \rangle \cdots \langle \alpha^N | \psi \rangle |\alpha^1\rangle \cdots |\alpha^N\rangle . \end{aligned} \quad (16)$$

* In this case it is also possible to define an operator which corresponds to the relative frequency: $\hat{F}_\alpha = \sum_{\alpha^1 \dots \alpha^N} |\alpha^1\rangle \cdots |\alpha^N\rangle f_\alpha(\alpha^1, \dots, \alpha^N) \langle \alpha^N| \cdots \langle \alpha^1|$ (See Ref. [8], [9], [12]).

Then from Eqs. (10), (11), (12), (14), (16) and Eq. (4.13) of Ref. [11], we can prove that

$$\begin{aligned}
\langle \chi_N^\epsilon | \chi_N^\epsilon \rangle &= \sum_{\substack{\alpha^1 \dots \alpha^N \\ \delta(\alpha^1 \dots \alpha^N) \geq \epsilon}} |\langle \alpha^1 | \psi \rangle|^2 \dots |\langle \alpha^N | \psi \rangle|^2 \\
&\leq \frac{1}{\epsilon} \sum_{\alpha^1 \dots \alpha^N} \delta(\alpha^1, \dots, \alpha^N) |\langle \alpha^1 | \psi \rangle|^2 \dots |\langle \alpha^N | \psi \rangle|^2 \\
&= \frac{1}{\epsilon} \sum_{\alpha^1 \dots \alpha^N} [f_\alpha(\alpha^1, \dots, \alpha^N) - |\langle \alpha | \psi \rangle|^2]^2 |\langle \alpha^1 | \psi \rangle|^2 \dots |\langle \alpha^N | \psi \rangle|^2 \quad (17) \\
&= \frac{1}{\epsilon} \sum_{\alpha} \frac{1}{N} |\langle \alpha | \psi \rangle|^2 (1 - |\langle \alpha | \psi \rangle|^2) \\
&\leq \frac{1}{N\epsilon} .
\end{aligned}$$

No matter how small we choose ϵ , we can always find an N big enough so that the norm of $|\chi_N^\epsilon\rangle$ becomes smaller than any positive number. This means that

$$\lim_{N \rightarrow \infty} |\Psi_N^\epsilon\rangle = |\Psi\rangle . \quad (18)$$

Therefore we have shown that $P[C_\alpha]$ are equal to the relative frequencies.

§ 3. Mixed State Case

In this section an individual system is a mixed state ρ :

$$\begin{aligned}
\rho &= \sum_i |\psi_i\rangle \pi_i \langle \psi_i| , \\
\sum_i \pi_i &= 1 , \quad \langle \psi_i | \psi_i \rangle = 1 . \quad (19)
\end{aligned}$$

The total system is written by the density matrix,

$$\rho^N = \sum_{i^1 \dots i^N} |\psi_{i^1}\rangle \dots |\psi_{i^N}\rangle \pi_{i^1} \dots \pi_{i^N} \langle \psi_{i^N} | \dots \langle \psi_{i^1} | . \quad (20)$$

Here upper indices of i distinguish the individual system. In the following discussion we assume the decoherence :

$$\begin{aligned}
D(C_{\alpha'}, C_\alpha) &= \sum_i \pi_i \langle \psi_i | C_\alpha^\dagger C_{\alpha'} | \psi_i \rangle \\
&= \delta_{\alpha\alpha'} \sum_i \pi_i \langle \psi_i | C_\alpha^\dagger C_\alpha | \psi_i \rangle , \quad (21)
\end{aligned}$$

where we have used Eqs. (4), (6), (19). We find

$$\begin{aligned}
P[C_\alpha] &= \sum_i \pi_i \langle \psi_i | C_\alpha^\dagger C_\alpha | \psi_i \rangle \\
&= \sum_i \pi_i \langle \psi_i | C_\alpha | \psi_i \rangle = \sum_i \pi_i \langle \psi_i | C_\alpha^\dagger | \psi_i \rangle ,
\end{aligned} \tag{22}$$

using Eqs. (2), (3), (8), (21). The Eqs. (10) hold in this case, too. We will show that $P[C_\alpha]$ are equal to the relative frequencies.

Defining

$$|\alpha, i\rangle = \frac{C_\alpha |\psi_i\rangle}{\sqrt{\sum_j \pi_j \langle \psi_j | C_\alpha | \psi_j \rangle}} , \tag{23}$$

we obtain from Eqs. (2), (3), (21), (22) that

$$\begin{aligned}
\sum_i \pi_i \langle \alpha, i | \beta, i \rangle &= \delta_{\alpha\beta} , \\
|\psi_i\rangle &= \sum_\alpha \sum_j \pi_j \langle \alpha, j | \psi_j \rangle |\alpha, i\rangle , \\
\left| \sum_i \pi_i \langle \alpha, i | \psi_i \rangle \right|^2 &= P[C_\alpha] .
\end{aligned} \tag{24}$$

We can expand ρ^N by these vectors $|\alpha, i\rangle$ and write

$$\begin{aligned}
\rho^N &= \sum_{i^1 \dots i^N} |\Psi_{i^1 \dots i^N}\rangle \pi_{i^1} \dots \pi_{i^N} \langle \Psi_{i^1 \dots i^N} | , \\
|\Psi_{i^1 \dots i^N}\rangle &= |\psi_{i^1}\rangle \dots |\psi_{i^N}\rangle \\
&= \sum_{\substack{\alpha^1 \dots \alpha^N \\ j^1 \dots j^N}} \pi_{j^1} \dots \pi_{j^N} \langle \alpha^1, j^1 | \psi_{j^1} \rangle \dots \langle \alpha^N, j^N | \psi_{j^N} \rangle \\
&\quad \times |\alpha^1, i^1\rangle \dots |\alpha^N, i^N\rangle .
\end{aligned} \tag{25}$$

Let us define

$$\begin{aligned}
|\Psi_{i^1 \dots i^N}^\epsilon\rangle &= \sum_{\substack{\alpha^1 \dots \alpha^N \\ \delta(\alpha^1 \dots \alpha^N) < \epsilon \\ j^1 \dots j^N}} \pi_{j^1} \dots \pi_{j^N} \langle \alpha^1, j^1 | \psi_{j^1} \rangle \dots \langle \alpha^N, j^N | \psi_{j^N} \rangle \\
&\quad \times |\alpha^1, i^1\rangle \dots |\alpha^N, i^N\rangle , \\
|\chi_{i^1 \dots i^N}^\epsilon\rangle &= \sum_{\substack{\alpha^1 \dots \alpha^N \\ \delta(\alpha^1 \dots \alpha^N) \geq \epsilon \\ j^1 \dots j^N}} \pi_{j^1} \dots \pi_{j^N} \langle \alpha^1, j^1 | \psi_{j^1} \rangle \dots \langle \alpha^N, j^N | \psi_{j^N} \rangle \\
&\quad \times |\alpha^1, i^1\rangle \dots |\alpha^N, i^N\rangle .
\end{aligned} \tag{26}$$

Again we can prove from Eqs. (10), (11), (12), (24), (26) and Eq. (4.13) of Ref. [11] that

$$\sum_{i^1 \dots i^N} \pi_{i^1} \dots \pi_{i^N} \langle \chi_{i^1 \dots i^N}^\epsilon | \chi_{i^1 \dots i^N}^\epsilon \rangle \leq \frac{1}{N\epsilon} . \tag{27}$$

Now if we assume

$$\langle \chi_{i^1 \dots i^N}^\epsilon | \chi_{i^1 \dots i^N}^\epsilon \rangle \geq c \quad (\exists c > 0) , \tag{28}$$

when $N \rightarrow \infty$, then Eq. (27) means

$$\sum_{i^1 \dots i^N} \pi_{i^1} \dots \pi_{i^N} c \leq \frac{1}{N\epsilon} \quad (N \rightarrow \infty) . \tag{29}$$

From Eqs. (19), (29) we obtain that

$$c \leq \frac{1}{N\epsilon} \quad (N \rightarrow \infty) , \tag{30}$$

but this is a contradiction to $c > 0$. Hence

$$\lim_{N \rightarrow \infty} \langle \chi_{i^1 \dots i^N}^\epsilon | \chi_{i^1 \dots i^N}^\epsilon \rangle = 0 . \tag{31}$$

This Eq. (31) means that in the expansion (25) we only need such α^I that satisfy $\delta(\alpha^1, \dots, \alpha^N) < \epsilon$, if we consider the limit of $N \rightarrow \infty$. So we have confirmed that $P[C_\alpha]$ are equal to the relative frequencies.

§ 4. Summary

In order to confirm the physical probability interpretation of the Gell-Mann and Hartle's generalized quantum theory, we started from N identical systems, which were pure

states (§2) or mixed states (§3). We found that the relative frequencies of histories C_α are equal to the diagonal elements of decoherence functional $P[C_\alpha]$, when the set of alternative histories decoheres and $N \rightarrow \infty$.

Acknowledgements

The author would like to thank Professor J.B. Hartle for valuable discussions and careful reading of the manuscript. He is also grateful to Professor G.T. Horowitz for helpful discussions. He would like to thank University of California, Santa Barbara for hospitality. This work was supported in part by Japanese Ministry of Education, Science and Culture.

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